TRAVELING WAVES FOR THE CUBIC SZEGÖ EQUATION ON THE REAL LINE

OANA POCOVNICU

Abstract. We consider the cubic Szegö equation

$$i\partial_t u = \Pi(|u|^2 u)$$

in the Hardy space $L^2_+(\mathbb{R})$ on the upper half-plane, where Π is the Szegö projector. It was first introduced by Gérard and Grellier in [5] as a toy model for totally non-dispersive evolution equations. We show that the only traveling waves are of the form $\frac{C}{x-p}$, $p \in \mathbb{C}$ with $\operatorname{Im} p < 0$. Moreover, they are shown to be orbitally stable, in contrast to the situation on the unit disk where some traveling waves were shown to be unstable.

1. Introduction

One of the most important properties in the study of the nonlinear Schrödinger equations (NLS) is *dispersion*. It is often exhibited in the form of the Strichartz estimates of the corresponding linear flow. In case of the cubic NLS:

$$i\partial_t u + \Delta u = |u|^2 u, \quad (t,x) \in \mathbb{R} \times M, \tag{1.1}$$

Burq, Gérard, and Tzvetkov [1] observed that the dispersive properties are strongly influenced by the geometry of the underlying manifold M. Taking this idea further, Gérard and Grellier [6] remarked that dispersion disappears completely when M is a sub-Riemannian manifold (for example, the Heisenberg group). In this situation, many of the classical arguments used in the study of NLS no longer hold. As a consequence, even the problem of global well-posedness of (1.1) on a sub-Riemannian manifold still remains open.

In [5, 6], Gérard and Grellier introduced a model of a non-dispersive Hamiltonian equation called *the cubic Szëgo equation.* (See (1.2) below.) The study of this equation is the first step toward understanding existence and other properties of smooth solutions of NLS in the absence of dispersion. Remarkably, the Szegö equation turned out to be completely integrable in the following sense. It possesses a Lax pair structure and an infinite sequence of conservation laws. Moreover, the dynamics can be approximated by a sequence of finite dimensional completely integrable Hamiltonian systems. To illustrate the degeneracy of this completely integrable structure, several instability phenomena were established in [5].

The Szëgo equation was studied in [5, 6] on the circle $\mathbb{S}^1$. More precisely, solutions were considered to belong at all time to the Hardy space $L^2_+(\mathbb{S}^1)$ on the unit disk $\mathbb{D} = \{|z| < 1\}$. This is the space of L^2-functions on $\mathbb{S}^1$ with $\hat{f}(k) = 0$ for all $k < 0$. These functions can be extended as holomorphic functions on the unit disk. Several properties of the Hardy space on the unit disk naturally transfer to the Hardy space $L^2_+(\mathbb{R})$ on the upper half-plane

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$\mathbb{C}_+ = \{z; \mathrm{Im} z > 0\}$, defined by

$$L^2_+(\mathbb{R}) = \Big\{f \text{ holomorphic on } \mathbb{C}_+; \|g\|_{L^2_+(\mathbb{R})} := \sup_{y>0} \left(\int_{\mathbb{R}} |g(x+iy)|^2 dx \right)^{1/2} < \infty \Big\}.$$

In view of the Paley-Wiener theorem, we identify this space of holomorphic functions on $\mathbb{C}_+$ with the space of its boundary values:

$$L^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}); \operatorname{supp} \hat{f} \subset [0,\infty)\}.$$

The transfer from $L^2_+(\mathbb{S}^1)$ to $L^2_+(\mathbb{R})$ is made by the usual conformal transformation $\omega : \mathbb{D} \to \mathbb{C}_+$ given by

$$\omega(z) = i\frac{1+z}{1-z}.$$

However, the image of a solution of the Szëgo equation on $\mathbb{S}^1$ under the conformal transformation is no longer a solution of the Szëgo equation on $\mathbb{R}$. Therefore, we directly study the Szëgo equation on $\mathbb{R}$ in the following.

Endowing $L^2(\mathbb{R})$ with the usual scalar product $(u,v) = \int_{\mathbb{R}} u\bar{v}$, we define the Szegö projector $\Pi : L^2(\mathbb{R}) \to L^2_+(\mathbb{R})$ to be the projector onto the non-negative frequencies,

$$\Pi(f)(x) = \frac{1}{2\pi} \int_0^\infty e^{ix\xi} \hat{f}(\xi) d\xi.$$

For $u \in L^2_+(\mathbb{R})$, we consider *the Szëgo equation on the real line*:

$$i\partial_t u = \Pi(|u|^2 u), \quad x \in \mathbb{R}. \tag{1.2}$$

This is a Hamiltonian evolution associated to the Hamiltonian

$$E(u) = \int_{\mathbb{R}} |u|^4 dx$$

defined on $L^4_+(\mathbb{R})$. From this structure, we obtain the formal conservation law $E(u(t)) = E(u(0))$. The invariance under translations and under modulations provides two more conservation laws, $Q(u(t)) = Q(u(0))$ and $M(u(t)) = M(u(0))$, where

$$Q(u) = \int_{\mathbb{R}} |u|^2 dx \quad \text{and} \quad M(u) = \int_{\mathbb{R}} \bar{u} D u \, dx, \quad \text{with } D = -i\partial_x.$$

Now, we define the Sobolev spaces $H^s_+(\mathbb{R})$ for $s \geq 0$:

$$H^s_+(\mathbb{R}) = \Big\{h \in L^2_+(\mathbb{R}); \|h\|_{H^s_+} := \left(\frac{1}{2\pi} \int_0^\infty (1+|\xi|^2)^s |\hat{h}(\xi)|^2 d\xi \right)^{1/2} < \infty \Big\}.$$

Similarly, we define the homogeneous Sobolev norm for $h \in \dot{H}^s_+$ by

$$||h||_{\dot{H}^s_+} := \left(\frac{1}{2\pi} \int_0^\infty |\xi|^{2s} |\hat{h}(\xi)|^2 \right)^{1/2} < \infty.$$

Slight modifications of the proof of the corresponding result in [5] lead to the following well-posedness result:

Theorem 1.1. *The cubic Szegö equation* (1.2) *is globally well-posed in $H^s_+(\mathbb{R})$ for $s \geq \frac{1}{2}$, i.e. given $u_0 \in H^{1/2}_+$, there exists a unique global-in-time solution $u \in C(\mathbb{R}; H^{1/2}_+)$ of* (1.2) *with initial condition u_0. Moreover, if $u_0 \in H^s_+$ for some $s > \frac{1}{2}$, then $u \in C(\mathbb{R}; H^s_+)$.*

In this paper, we concentrate on the study of traveling waves. The two main goals are the classification of traveling waves and their stability. As a result, we show that the situation on the real line is essentially different from that on the circle.

A solution for the cubic Szegö equation on the real line (1.2) is called a *traveling wave* if there exist $c, \omega \in \mathbb{R}$ such that

$$u(t,z) = e^{-i\omega t} u_0(z-ct), \quad z \in \mathbb{C}_+ \cup \mathbb{R}, t \in \mathbb{R} \tag{1.3}$$

for some $u_0 \in H^{1/2}_+(\mathbb{R})$. Note that a solution to (1.2) in $H^{1/2}_+(\mathbb{R})$ has a natural extension onto $\mathbb{C}_+$, and we have used this viewpoint in (1.3). Substituting (1.3) into (1.2), we obtain that u_0 satisfies the following equation on $\mathbb{R}$:

$$cDu_0 + \omega u_0 = \Pi(|u_0|^2 u_0). \tag{1.4}$$

In the following, we use the simpler notation u instead of u_0, when we study time-independent problems. From (1.4), we see that traveling waves with nonzero velocity, $c \neq 0$, have good regularity. Indeed, we prove that $u \in H^s_+(\mathbb{R})$ for all $s \geq 0$ in Lemma 3.1. In particular, by Sobolev embedding theorem, we have $u \in L^p_+(\mathbb{R})$ for $2 \leq p \leq \infty$. On the other hand, equation (1.4) yields in Lemma 4.1 that there exist no nontrivial stationary waves, i.e. traveling waves of velocity $c = 0$, in L^2_+.

Now, we present our main results:

Theorem 1.2. *A function $u \in C(\mathbb{R}, H^{1/2}_+(\mathbb{R}))$ is a traveling wave if and only if there exist $C, p \in \mathbb{C}$ with $\operatorname{Im} p < 0$ such that*

$$u(0,z) = \frac{C}{z-p}. \tag{1.5}$$

Theorem 1.3. *Let $a > 0$, $r > 0$, and consider the cylinder*

$$C(a,r) = \Big\{ \frac{\alpha}{z-p}; |\alpha| = a, \ \operatorname{Im} p = -r \Big\}.$$

Let $\{u^n_0\} \subset H^{1/2}_+$ with

$$\inf_{\phi \in C(a,r)} \|u^n_0 - \phi\|_{H^{1/2}_+} \to 0 \text{ as } n \to +\infty,$$

and let u^n denote the solution to (1.2) with initial data u^n_0. Then

$$\sup_{t\in\mathbb{R}} \inf_{\phi \in C(a,r)} \|u^n(t,x) - \phi(x)\|_{H^{1/2}_+} \to 0.$$

Let us compare our results to those obtained in [5]. In the case of the Szegö equation on $\mathbb{S}^1$, the nontrivial stationary waves ($c = 0$) are finite Blaschke products of the form

$$\alpha \prod_{j=1}^{N} \frac{z - p_j}{1 - p_j z},$$

where $|\alpha|^2 = \omega$, $N \in \mathbb{N}$, and $p_1, p_2, ..., p_N \in \mathbb{D}$, and the traveling waves with nonzero velocity are rational fractions of the form:

$$\frac{Cz^l}{z^N - p}, \tag{1.6}$$

where $N \in \mathbb{N}$, $l \in \{0, 1, \dots, N-1\}$, $C, p \in \mathbb{C}$, and $|p| > 1$. Moreover, instability phenomena were displayed for some of the above traveling waves. For the cubic Szegö equation on $\mathbb{R}$, Theorems 1.2 and 1.3 state that there exist less traveling waves (corresponding to $N = 1$ and $l = 0$ in (1.6)) and that there is no instability phenomenon.

The proof of Theorem 1.2 involves arguments from several areas of analysis: a Kronecker-type theorem, scattering theory, existence of a Lax pair structure, a theorem by Lax on invariant subspaces of the Hardy space, and canonical factorization of Beurling-Lax inner functions. In the following, we introduce the main notions and known results, and briefly describe the strategy of the proof.

As in [5], an important property of the Szëgo equation on $\mathbb{R}$ is the existence of a Lax pair structure. Using the Szegö projector, we first define two important classes of operators on L^2_+: *the Hankel and Toeplitz operators.* We use these operators to find a Lax pair. See Proposition 1.4.

A Hankel operator $H_u : L^2_+ \to L^2_+$ of symbol $u \in H^{1/2}_+$ is defined by

$$H_u(h) = \Pi(u\bar{h}).$$

Note that H_u is $\mathbb{C}$-antilinear and satisfies

$$(H_u(h_1), h_2) = (H_u(h_2), h_1). \tag{1.7}$$

In Lemma 3.5 below we prove that H_u is a Hilbert-Schmidt operator of Hilbert-Schmidt norm $\frac{1}{\sqrt{2\pi}}\|u\|_{\dot{H}^{1/2}}$.

A Toeplitz operator $T_b : L^2_+ \to L^2_+$ of symbol $b \in L^\infty(\mathbb{R})$ is defined by

$$T_b(h) = \Pi(bh).$$

T_b is $\mathbb{C}$-linear. Moreover, T_b is self-adjoint if and only if b is real-valued.

Proposition 1.4. *Let $u \in C(\mathbb{R}; H^s_+)$ for some $s > \frac{1}{2}$. The cubic Szegö equation* (1.2) *is equivalent to the following evolution equation:*

$$\frac{d}{dt}H_u = [B_u, H_u], \tag{1.8}$$

where $B_u = \frac{i}{2}H_u^2 - iT_{|u|^2}$. In other words, the pair (H_u, B_u) is a Lax pair for the cubic Szegö equation on the real line.

The proof of Proposition 1.4 follows the same lines as that of the corresponding result on $\mathbb{S}^1$ in [5], and is based on the following identity:

$$H_{\Pi(|u|^2u)} = T_{|u|^2}H_u + H_uT_{|u|^2} - H_u^3. \tag{1.9}$$

Combining (1.4) and (1.9), we deduce that if u is a traveling wave with $c \neq 0$, then the following identity holds:

$$A_uH_u + H_uA_u + \frac{\omega}{c}H_u + \frac{1}{c}H_u^3 = 0, \tag{1.10}$$

where

$$A_u = D - \frac{1}{c}T_{|u|^2}. \tag{1.11}$$

In Section 2, we prove a Kronecker-type theorem for the Hardy space $L^2_+(\mathbb{R})$, where we classify all the symbols u such that the operator H_u has finite rank. For a proof of the classical theorem for $L^2_+(\mathbb{S}^1)$, due to Kronecker, see [5].

We prove Theorem 1.2 in Section 4. We first prove that all the traveling waves are rational fractions. On $\mathbb{S}^1$, this follows easily from the Kronecker theorem and the fact that the operator A_u has discrete spectrum. On $\mathbb{R}$, however, it turns out that A_u has continuous spectrum. Therefore, we use scattering theory to study the spectral properties of A_u in

detail in Section 3. More precisely, we show that the generalized wave operators $\Omega^{\pm}(D, A_u)$, rigorously defined by (3.1) below, exist and are complete. As a result, we obtain that

$$\mathcal{H}_{\mathrm{ac}}(A_u) \subset \operatorname{Ker} H_u,$$

where $\mathcal{H}_{\mathrm{ac}}(A_u)$ is the absolutely continuous subspace of A_u. The subspace $\operatorname{Ker} H_u$ plays an important role in our analysis. More precisely, it turns out to be invariant under multiplication by $e^{i\alpha x}$, for all $\alpha \geq 0$. Therefore, applying a theorem by Lax (Proposition 4.4 below) on invariant subspaces, it results that

$$\operatorname{Ker} H_u = \phi L^2_+,$$

where ϕ is an inner function in the sense of Beurling-Lax, i.e. a bounded holomorphic function on $\mathbb{C}_+$ such that $|\phi(x)| = 1$ for all $x \in \mathbb{R}$. Using the Lax pair structure and the identity (1.10), we show that ϕ satisfies the following simple equation:

$$cD\phi = |u|^2\phi.$$

However, as an inner function, ϕ satisfies a canonical factorization (4.3). From this, it follows that ϕ belongs to a special class of inner functions, the finite Blaschke products, i.e.

$$\phi(z) = \prod_{j=1}^{N} \frac{z - \lambda_j}{z - \overline{\lambda_j}},$$

where $N \in \mathbb{N}$ and $\mathrm{Im}\lambda_j > 0$ for all $j = 1, 2, \dots, N$. The Kronecker-type theorem then yields that the traveling wave u is a rational fraction.

In the case of $\mathbb{S}^1$, the natural shift, multiplication by e^{ix}, was used in concluding traveling waves are of the form (1.6). In our case, we use the "infinitesimal" shift, multiplication by x, to show that traveling waves are of the form (1.5).

Finally, we prove Theorem 1.3 in Section 5. The orbital stability of traveling waves is a consequence of the fact that traveling waves are ground states for the following inequality, an analogue of Weinstein's sharp Gagliardo-Nirenberg inequality in [17].

Proposition 1.5. *For all $u \in H^{1/2}_+(\mathbb{R})$ the following Gagliardo-Nirenberg inequality holds:*

$$\|u\|_{L^4} \leq \frac{1}{\sqrt[4]{\pi}} \|u\|_{L^2}^{1/2} \|u\|_{\dot{H}^{1/2}_+}^{1/2}, \tag{1.12}$$

or, equivalently,

$$E \leq \frac{1}{\pi} MQ.$$

Moreover, equality holds if and only if $u = \frac{C}{x-p}$, where $C, p \in \mathbb{C}$ with $\mathrm{Im}p < 0$.

Remark 1.6. As a consequence of Proposition 1.5, one can verify that the functions $u = \frac{C}{x-p}$, with $\mathrm{Im}p < 0$, are indeed initial data for traveling waves. More precisely, since they are minimizers of the functional

$$v \in H^{1/2}_+ \mapsto M(v)Q(v) - \pi E(v),$$

the differential of this functional at u is zero. Thus,

$$\frac{1}{2}Q(u)Du + \frac{1}{2}M(u)u - \pi\Pi(|u|^2u) = 0.$$

Consequently, u is a solution of equation (1.4) with

$$c = \frac{Q(u)}{2\pi} = \frac{|C|^2}{-2\mathrm{Im}p}, \qquad \omega = \frac{M(u)}{2\pi} = \frac{|C|^4}{4(-\mathrm{Im}p)^3}$$

and hence it is an initial datum for a traveling wave.

In the case of $\mathbb{S}^1$, the Gagliardo-Nirenberg inequality suffices to conclude the stability of the traveling waves with $N = 1$. However, in the case of $\mathbb{R}$, we need to use in addition a concentration-compactness argument. This concentration-compactness argument, which first appeared in the work of Cazenave and Lions [2], was refined and turned into profile decomposition theorems by Gérard [4] and later by Hmidi and Keraani [7]. We use it in the form of Proposition 5.1, a profile decomposition theorem for bounded sequences in $H^{1/2}_+$.

We conclude this introduction by presenting two open problems. Here, we use the term soliton instead of traveling wave, so that we put into light several connections with existing works. The first problem is the soliton resolution, which consists in writing any solution as a superposition of solitons and radiation. For the KdV equation, this property was rigorously stated in [3] for initial data to which the Inverse Scattering Transform applies. Therefore, for the Szëgo equation, one needs to solve inverse spectral problems for the Hankel operators and also find explicit action angle coordinates.

The second open problem is the interaction of solitons with external potentials. Consider the Szëgo equation with a linear potential, where initial data are taken to be of the form (1.5). As in the works of Holmer and Zworski [9] and Perelman [14], it would be interesting to investigate if solutions of the perturbed Szëgo equation can be approximated by traveling wave solutions to the original Szegö equation (1.2).

2. A Kronecker-type theorem

A theorem by Kronecker asserts in the setting of $\mathbb{S}^1$ that the set of symbols u such that H_u is of rank N is precisely a $2N$-dimensional complex submanifold of $L^2_+(\mathbb{S}^1)$ containing only rational fractions. In this section, we prove the analogue of this theorem in the case of $L^2_+(\mathbb{R})$. For a different proof of a similar result on some Hankel operators on $L^2_+(\mathbb{R})$ defined in a slightly different way, we refer to Lemma 8.12, p.54 in [13].

Definition 1. *Let $N \in \mathbb{N}^*$. We denote by $\mathcal{M}(N)$ the set of rational fractions of the form*

$$\frac{A(z)}{B(z)},$$

where $A \in \mathbb{C}_{N-1}[z]$, $B \in \mathbb{C}_N[z]$, $0 \leq \deg(A) \leq N-1$, $\deg(B) = N$, $B(0) = 1$, $B(z) \neq 0$, for all $z \in \mathbb{C}_+ \cup \mathbb{R}$, and A and B have no common factors.

Theorem 2.1. *The function u belongs to $\mathcal{M}(N)$ if and only if the Hankel operator H_u has complex rank N.*

Moreover, if $u \in \mathcal{M}(N)$, $u(z) = \frac{A(z)}{B(z)}$, where $B(z) = \prod_{j=1}^J (z-p_j)^{m_j}$, with $\sum_{j=1}^J m_j = N$ and $\operatorname{Im} p_j < 0$ for all $j = 1, 2, ..., J$, then the range of H_u is given by

$$\operatorname{Ran} H_u = \operatorname{span}_{\mathbb{C}} \left\{ \frac{1}{(z-p_j)^m}, 1 \leq m \leq m_j \right\}_{j=1}^J \tag{2.1}$$

Proof. The theorem will follow once we prove:

(i) $u \in \mathcal{M}(N) \Longrightarrow \operatorname{rk}(H_u) \leq N$

(ii) $\operatorname{rk}(H_u) = N \Longrightarrow u \in \mathcal{M}(N)$.

Let us first prove (i). Let $u \in \mathcal{M}(N)$, i.e. u is a linear combination of

$$\frac{1}{(z-p)^m},$$

where Im$p<0$, $1\le m\le m_p$, and $\sum m_p = N$. Then, computing the integral

$$\int_{\mathbb{R}} \frac{e^{-ix\xi}}{(x-p)^m}dx,$$

using the residue theorem, we obtain that $\hat{u}(\xi)=0$ for all $\xi\le 0$ and $\hat{u}(\xi)$ is a linear combination of $\xi^{m-1}e^{-ip\xi}$, with $1\le m\le m_p$, for $\xi>0$.

Given $h\in L^2_+$, we have $\widehat{H_u(h)}(\xi)=0$ for $\xi<0$. Moreover, for $\xi>0$, we have

$$\begin{aligned}
\widehat{H_u(h)}(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{0}\hat{u}(\xi-\eta)\hat{\bar{h}}(\eta)d\eta \\
&= \frac{1}{2\pi}\int_0^{\infty}\hat{u}(\xi+\eta)\overline{\hat{h}}(\eta)d\eta \\
&= \sum_{\substack{1\le m\le m_p\\ \sum m_p=N}} c_{m,p}\Big(\sum_{k=0}^{m-1}C_{m-1}^k\xi^{m-1-k}\int_0^{\infty}\eta^k\overline{\hat{h}}(\eta)e^{-ip\eta}d\eta\Big)e^{-ip\xi} \\
&= \sum_{\substack{1\le m\le m_p\\ \sum m_p=N}} \tilde{d}_{m,p}(u,h)\xi^{m-1}e^{-ip\xi} = \sum_{\substack{1\le m\le m_p\\ \sum m_p=N}} d_{m,p}(u,h)\Big(\frac{1}{(x-p)^m}\Big)^{\wedge}(\xi),
\end{aligned} \tag{2.2}$$

where $c_{m,p}$, $\tilde{d}_{m,p}$, $d_{m,p}$ are constants depending on p and m.

Hence,

$$H_u(h)(x) = \sum_{\substack{1\le m\le m_p\\ \sum m_p=N}}\frac{d_{m,p}(u,h)}{(x-p)^m} \tag{2.3}$$

and $\mathrm{rk}(H_u)\le N$.

Let us now prove (ii). Assume that $\mathrm{rank}(H_u)=N$, i.e. the range of H_u, $\mathrm{Ran}\,H_u$, is a $2N$-dimensional real vector space. As H_u is $\mathbb{C}$-antilinear, one can choose a basis of $\mathrm{Ran}\,H_u$ of eigenvectors of H_u in the following way:

$$\{v_1, iv_1, ..., v_N, iv_N\,;\, H_u(v_j)=\lambda_j v_j, \lambda_j>0, j=1,2,\dots,N\}$$

Let $w_j=\sqrt{\lambda_j}v_j$. If $h\in L^2_+$, then by Parseval's identity we have

$$\begin{aligned}
H_u(h) &= \sum_{j=1}^N (H_u(h),v_j)v_j + \sum_{j=1}^N (H_u(h),iv_j)iv_j = 2\sum_{j=1}^N (H_u(h),v_j)v_j = 2\sum_{j=1}^N (H_u(v_j),h)v_j \\
&= 2\sum_{j=1}^N(\lambda_j v_j,h)v_j = 2\sum_{j=1}^N (w_j,h)w_j = \frac{1}{\pi}\sum_{j=1}^N\Big(\int_0^{\infty}\hat{w}_j(\eta)\overline{\hat{h}}(\eta)d\eta\Big)w_j.
\end{aligned}$$

Consequently,

$$\widehat{H_u(h)}(\xi) = \frac{1}{2\pi}\mathbf{1}_{\xi\ge 0}\int_0^{\infty}\hat{u}(\xi+\eta)\overline{\hat{h}}(\eta)d\eta = \frac{1}{\pi}\mathbf{1}_{\xi\ge 0}\sum_{j=1}^N\int_0^{\infty}\hat{w}_j(\eta)\hat{w}_j(\xi)\overline{\hat{h}}(\eta)d\eta.$$

and hence,

$$\mathbf{1}_{\xi\ge 0}\int_0^{\infty}\Big(\hat{u}(\xi+\eta)-2\sum_{j=1}^N\hat{w}_j(\eta)\hat{w}_j(\xi)\Big)\overline{\hat{h}}(\eta)d\eta = 0,$$

for all $h \in L^2_+$. Therefore, for all $\xi, \eta \geq 0$, we have

$$\hat{u}(\xi+\eta) = 2\sum_{j=1}^{N} \hat{w}_j(\eta)\hat{w}_j(\xi). \tag{2.4}$$

Let $L > 2N+1$ be an even integer and ϕ be the probability density function of the chi-square distribution defined by

$$\phi(\xi) = \begin{cases} \frac{1}{2^{\frac{L}{2}}\Gamma(\frac{L}{2})}\xi^{\frac{L}{2}-1}e^{-\frac{\xi}{2}}, \text{ if } \xi \geq 0 \\ 0, \text{ if } \xi < 0, \end{cases}$$

where Γ is the Gamma function. Then, its Fourier transform is

$$\widehat{\phi}(x) = (1+2ix)^{-\frac{L}{2}}.$$

Notice that $\phi \in H^N(\mathbb{R})$ since

$$\|\phi\|^2_{H^N} = \int_{\mathbb{R}} \frac{\langle x\rangle^{2N}}{|1+2ix|^L} dx$$

which is convergent if and only if $2N - L < -1$.

Let $\langle \theta, \psi\rangle = \int_{\mathbb{R}} \theta(x)\psi(x)$ for all $\theta \in H^{-N}(\mathbb{R})$ and $\psi \in H^N(\mathbb{R})$. Consider the matrix A_ϕ defined by:

$$\begin{pmatrix} \langle \hat{w}_1, \phi\rangle & \langle \hat{w}_1', \phi\rangle & \cdots & \langle \hat{w}_1^{(N)}, \phi\rangle \\ \langle \hat{w}_2, \phi\rangle & \langle \hat{w}_2', \phi\rangle & \cdots & \langle \hat{w}_2^{(N)}, \phi\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \hat{w}_N, \phi\rangle & \langle \hat{w}_N', \phi\rangle & \cdots & \langle \hat{w}_N^{(N)}, \phi\rangle \end{pmatrix}$$

Since $\mathrm{rk}(A_\phi) \leq N$, it results that there exists $(c_0, c_1, \dots, c_N) \neq 0$ such that

$$\Big\langle \sum_{k=0}^{N} c_k \hat{w}_j{}^{(k)}, \phi \Big\rangle = 0,$$

for all $j = 1, 2, \dots, N$. Then, since $\operatorname{supp}\phi \subset [0,\infty)$ and by (2.4), we have for all $\eta \geq 0$ that

$$\begin{aligned} \sum_{k=0}^{N} \Big\langle c_k \hat{u}^{(k)}(\xi), \phi(\xi-\eta)\Big\rangle_\xi &= \sum_{k=0}^{N} \Big\langle c_k \hat{u}^{(k)}(\xi+\eta), \phi(\xi)\Big\rangle_\xi = \sum_{k=0}^{N}(-1)^k c_k \int_0^\infty \hat{u}(\xi+\eta)\phi^{(k)}(\xi)d\xi \\ &= 2\sum_{k=0}^{N}(-1)^k c_k \int_0^\infty \Big(\sum_{j=1}^{N} \hat{w}_j(\eta)\hat{w}_j(\xi)\Big)\phi^{(k)}(\xi)d\xi \\ &= 2\sum_{j=1}^{N} \hat{w}_j(\eta) \sum_{k=0}^{N} c_k \Big\langle \hat{w}_j^{(k)}(\xi), \phi(\xi)\Big\rangle = 0. \end{aligned}$$

Denote $T = \sum_{k=0}^{N} c_k \hat{u}^{(k)}$. Then $T \in H^{-N}$ and $\operatorname{supp} T \in [0,\infty)$. We have just proved that for all $\eta \geq 0$

$$\begin{aligned} 0 = \langle T, \phi(\cdot - \eta)\rangle &= \int_{\mathbb{R}} T(\xi)\phi(\xi-\eta)d\xi = \int_{\mathbb{R}} T(\xi)\Big(\int_{\mathbb{R}} \frac{e^{ix(\xi-\eta)}}{(1+2ix)^{L/2}}dx\Big)d\xi \\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} T(\xi)e^{ix\xi}d\xi\Big)\frac{e^{-ix\eta}}{(1+2ix)^{L/2}}dx = \int_{\mathbb{R}} \mathcal{F}^{-1}T(x)\frac{e^{-ix\eta}}{(1+2ix)^{L/2}}dx. \end{aligned}$$

Denoting $R(x) := \frac{1}{(1+2ix)^{L/2}}\mathcal{F}^{-1}T(x)$, we have $\hat{R} \in H^{L/2-N}(\mathbb{R}) \subset H^{1/2}(\mathbb{R})$ and

$$0 = \int_{\mathbb{R}} R(x)e^{-ix\eta}dx = \hat{R}(\eta), \text{ for all } \eta \geq 0.$$

Thus $\operatorname{supp}\hat{R} \subset (-\infty, 0]$. By the definition of R, $(1-2D_\xi)^{L/2}\hat{R}(\xi) = T(\xi)$. Since the left hand-side is supported on $(-\infty, 0]$ and the right hand-side is supported on $[0,\infty)$, we deduce that $\operatorname{supp} T \subset 0$. In particular, $T_{|\xi>0} = 0$. This yields that $\hat{u}_{|\xi>0}$ is a weak solution on $(0,\infty)$ of the following linear ordinary differential equation:

$$\sum_{k=0}^{N} c_k v^{(k)}(\xi) = 0. \tag{2.5}$$

Then, by [8, Theorem 4.4.8, p.115], we have that $\hat{u}_{|\xi>0} \in C^N((0,\infty))$, $\hat{u}_{|\xi>0}$ is a classical solution of this equation and therefore it is a linear combination of

$$\xi^{m-1}e^{q\xi}$$

where $q \in \mathbb{C}$ is a root of the polynomial $P(X) = \sum_{k=0}^{N} c_k X^k$ with multiplicity m_q, $1 \leq m \leq m_q$, and $\sum_q m_q = N$. Note that we must have $\operatorname{Re} q < 0$, because $u \in L^2_+(\mathbb{R})$. Therefore we will denote $q = -ip$, with $\operatorname{Im} p < 0$ and obtain that $\hat{u}(\xi)$ is a linear combination of $\xi^{m-1}e^{-ip\xi}$ for $\xi > 0$. By the hypothesis $u \in L^2_+(\mathbb{R})$, we obtain $\hat{u}(\xi) = 0$ for $\xi \leq 0$. Hence for all $\xi \in \mathbb{R}$, $\hat{u}(\xi)$ is a linear combination of $\left(\frac{1}{(x-p)^m}\right)^{\wedge}(\xi)$, with $1 \leq q \leq m_q$ and $\sum m_q = N$. Thus $u \in \mathcal{M}(N')$ for some $N' \leq N$. If $N' < N$, then (i) yields $\operatorname{rk}(H_u) \leq N'$, which contradicts our assumption. In conclusion $u \in \mathcal{M}(N)$.

Finally, when $u \in \mathcal{M}(N)$ we have $\operatorname{rk}(H_u) = N$ and equation (2.3), and thus (2.1) follows. □

As a consequence of (2.1) we make the following remark.

Remark 2.2. *If $u \in \mathcal{M}(N)$, then $u \in \operatorname{Ran} H_u$.*

3. Spectral properties of the operator A_u for a traveling wave u

Let us first recall the definition and the basic properties of the generalized wave operators, which are the main objects in scattering theory. We refer to chapter XI in [15] for more details.

Let A and B be two self-adjoint operators on a Hilbert space $\mathcal{H}$. The basic principle of scattering theory is to compare the free dynamics corresponding to e^{-iAt} and e^{-iBt}. The fact that $e^{-iBt}\phi$ "looks asymptotically free" as $t \to -\infty$, with respect to A, means that there exists $\phi_+ \in \mathcal{H}$ such that

$$\lim_{t\to-\infty} \|e^{-iBt}\phi - e^{-itA}\phi_+\| = 0$$

or equivalently,

$$\lim_{t\to-\infty} \|e^{iAt}e^{-itB}\phi - \phi_+\| = 0.$$

Hence, we reduced ourselves to the problem of the existence of a strong limit. Let $\mathcal{H}_{\text{ac}}(B)$ be the absolutely continuous subspace for B and let $P_{\text{ac}}(B)$ be the orthogonal projection onto this subspace. In the definition of the generalized wave operators we have $\phi \in \mathcal{H}_{\text{ac}}(B)$.

We say that the generalized wave operators exist if the following strong limits exist:

$$\Omega^{\pm}(A,B) = \lim_{t\to\mp\infty} e^{itA}e^{-itB}P_{\mathrm{ac}}(B). \tag{3.1}$$

The wave operators $\Omega^{\pm}(A,B)$ are partial isometries with initial subspace $\mathcal{H}_{\mathrm{ac}}(B)$ and with values in Ran $\Omega^{\pm}(A,B)$. Moreover, $\operatorname{Ran}\Omega^{\pm}(A,B) \subset \mathcal{H}_{\mathrm{ac}}(A)$. If $\operatorname{Ran}\Omega^{\pm}(A,B) = \mathcal{H}_{\mathrm{ac}}(A)$, we say that the generalized wave operators are complete.

Lastly, we note that the following equality holds:

$$A\Omega^{\pm}(A,B) = \Omega^{\pm}(A,B)B. \tag{3.2}$$

Lemma 3.1. *If $u \in H^{1/2}_+$ is a traveling wave, then $u \in H^s_+(\mathbb{R})$ for all $s \geq 0$. In particular, by Sobolev embedding theorem, we have $u \in L^p(\mathbb{R})$ for $2 \leq p \leq \infty$.*

Proof. Because $u \in H^{1/2}(\mathbb{R})$, the Sobolev embedding theorem yields $u \in L^p(\mathbb{R})$, for all $2 \leq p < \infty$. Therefore $|u|^2u \in L^2(\mathbb{R})$ and thus $\Pi(|u|^2u) \in L^2_+$. Using equation (1.4)

$$cDu + \omega u = \Pi(|u|^2u),$$

we deduce that $Du \in L^2_+$. Consequently, $u \in H^1_+$ and by Sobolev embedding theorem we have $u \in L^\infty(\mathbb{R})$. Then $u^2D\bar{u}, |u|^2Du \in L^2(\mathbb{R})$. Applying the operator D to both sides of equation (1.4), we obtain $D^2u \in L^2(\mathbb{R})$ and hence $u \in H^2_+$. Iterating this argument infinitely many times, the conclusion follows. $\square$

Proposition 3.2. *Let u be a traveling wave. Then, $(A_u + i)^{-1} - (D+i)^{-1}$ is a trace class operator.*

Proof. We prove first that for all $f \in L^2(\mathbb{R})$, the operator $(D+i)^{-1}f$, defined on $L^2(\mathbb{R})$ by

$$\big((D+i)^{-1}f\big)h(x) = (D+i)^{-1}(fh)(x)$$

is Hilbert-Schmidt. Denote by $\mathcal{F}$ the Fourier transform. Using the isomorphism of $L^2(\mathbb{R})$ induced by the Fourier transform, we have that $(D+i)^{-1}f$ is a Hilbert-Schmidt operator if and only if $\mathcal{F}(D+i)^{-1}f$ is a Hilbert-Schmidt operator. The latter is an integral operator of kernel $K(\xi,\eta) = \frac{1}{2\pi}\cdot\frac{1}{\xi+i}\hat{f}(\xi-\eta)$. Indeed,

$$\mathcal{F}\big((D+i)^{-1}fh\big)(\xi) = \frac{1}{2\pi}\cdot\frac{1}{\xi+i}\widehat{fh}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{\xi+i}\hat{f}(\xi-\eta)\hat{h}(\eta)d\eta = \int_{\mathbb{R}} K(\xi,\eta)\hat{h}(\eta)d\eta.$$

Therefore, it is Hilbert-Schmidt if and only if $K(\xi,\eta) \in L^2_{\xi,\eta}(\mathbb{R}\times\mathbb{R})$. By the change of variables $\eta \mapsto \zeta = \xi - \eta$ we have

$$\|K(\xi,\eta)\|^2_{L^2_{\xi,\eta}} = \frac{1}{4\pi^2}\int_{\mathbb{R}}\frac{d\xi}{\xi^2+1}\int_{\mathbb{R}}|\hat{f}(\zeta)|^2d\zeta = C\|f\|^2_{L^2} < \infty.$$

Hence $(D+i)^{-1}f$ is a Hilbert-Schmidt operator and so is $\bar{f}(D+i)^{-1}$, its adjoint. According to Lemma 3.1, $u \in L^\infty(\mathbb{R})$ and thus $|u|^2 \in L^2(\mathbb{R})$. Taking $f = |u|^2$ and $f = u$, we conclude that the operators $(D+i)^{-1}|u|^2$, $(D+i)^{-1}u$, and $\bar{u}(D+i)^{-1}$ are all Hilbert-Schmidt.

We write

$$\begin{aligned}(A_u+i)^{-1} - (D+i)^{-1} &= (D+i)^{-1}(D-A_u)(A_u+i)^{-1}\\ &= \frac{1}{c}(D+i)^{-1}T_{|u|^2}(A_u+i)^{-1}\\ &= \frac{1}{c}\Pi(D+i)^{-1}|u|^2(A_u+i)^{-1} = L(A_u+i)^{-1},\end{aligned}$$

where $L = \frac{1}{c}\Pi(D+i)^{-1}|u|^2$. Note that L is a Hilbert-Schmidt operator since it is the composition of the bounded operator $\frac{1}{c}\Pi : L^2(\mathbb{R}) \to L^2_+$ with the Hilbert-Schmidt operator $(D+i)^{-1}|u|^2$. Finally, we write, using the latter formula twice

$$\begin{aligned}(A_u+i)^{-1} - (D+i)^{-1} &= L(L(A_u+i)^{-1} + (D+i)^{-1}) \\ &= L \circ L \circ (A_u+i)^{-1} + \frac{1}{c}\Pi(D+i)^{-1}u \circ \bar{u}(D+i)^{-1}.\end{aligned}$$

We obtain that $(A_u+i)^{-1} - (D+i)^{-1}$ is a trace class operator since the composition of two Hilbert-Schmidt operators is a trace class operator. □

Corollary 3.3. *If u is a traveling wave, then the wave operators $\Omega^{\pm}(D, A_u)$ exist and are complete.*

Proof. This easily follows from Kuroda-Birman theorem that we state below [15, Theorem XI.9]:

Let A and B be two self-adjoint operators on a Hilbert space such that $(A+i)^{-1}-(B+i)^{-1}$ is a trace class operator. Then $\Omega^{\pm}(A, B)$ exist and are complete. □

Corollary 3.4. *If u is a traveling wave, then $\sigma_{\mathrm{ac}}(A_u) = [0, +\infty)$.*

Proof. Since $\Omega^{\pm}(D, A_u)$ are complete, it results that they are isometries from $\mathcal{H}_{\mathrm{ac}}(A_u)$ onto $\mathcal{H}_{\mathrm{ac}}(D) = L^2_+$. By (3.2), we then have

$$A_{u|_{\mathcal{H}_{\mathrm{ac}}(A_u)}} = [\Omega^{\pm}(D, A_u)_{|_{\mathcal{H}_{\mathrm{ac}}(A_u)}}]^{-1} D \Omega^{\pm}(D, A_u)_{|_{\mathcal{H}_{\mathrm{ac}}(A_u)}}.$$

Consequently, $\sigma_{\mathrm{ac}}(A_u) = \sigma_{\mathrm{ac}}(D) = [0, +\infty)$. □

Our main goal in the following is to prove $\mathcal{H}_{\mathrm{ac}}(A_u) \subset \operatorname{Ker} H_u$. As we see below, it is enough to prove that $\left[\Omega^+(D, A_u)H_u^2\right](\mathcal{H}_{\mathrm{ac}}(A_u)) = 0$.

Lemma 3.5. *The operator H_u is a Hilbert-Schmidt operator on $L^2_+(\mathbb{R})$ of Hilbert-Schmidt norm $\frac{1}{\sqrt{2\pi}}\|u\|_{\dot{H}^{1/2}_+}$.*

Proof. Let us denote by $\|T\|_{HS}$ the Hilbert-Schmidt norm of a Hilbert-Schmidt operator T. By (2.2), we have

$$\widehat{H_u(h)}(\xi) = \frac{1}{2\pi}\mathbf{1}_{\xi \geq 0} \int_0^{\infty} \hat{u}(\xi+\eta)\overline{\hat{h}}(\eta)d\eta.$$

Then, we obtain

$$\begin{aligned}H_u(h)(x) &= \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty} e^{ix\xi}\hat{u}(\xi+\eta)\overline{\hat{h}}(\eta)d\eta d\xi \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}}\left(\int_0^{\infty}\int_0^{\infty} e^{ix\xi}e^{iy\eta}\hat{u}(\xi+\eta)d\eta d\xi\right)\bar{h}(y)dy.\end{aligned}$$

Using the fact that the Hilbert-Schmidt norm of an operator is equal to the norm of its integral kernel, Plancherel's formula, and Fubini's theorem, we have

$$\begin{aligned}\|H_u(h)\|^2_{HS} &= \frac{1}{16\pi^4}\left\|\int_0^{\infty}\int_0^{\infty} e^{ix\xi}e^{iy\eta}\hat{u}(\xi+\eta)d\eta d\xi\right\|^2_{L^2_{x,y}} = \frac{1}{4\pi^2}\|\mathbf{1}_{\xi\geq 0}\mathbf{1}_{\eta \geq 0}\hat{u}(\xi+\eta)\|^2_{L^2_{\eta,\xi}} \\ &= \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty}|\hat{u}(\xi+\eta)|^2 d\eta d\xi = \frac{1}{4\pi^2}\int_0^{\infty}\int_{\xi}^{\infty}|\hat{u}(\zeta)|^2 d\zeta d\xi \\ &= \frac{1}{4\pi^2}\int_0^{\infty}\left(\int_0^{\zeta} d\xi\right)|\hat{u}(\zeta)|^2 d\zeta = \frac{1}{4\pi^2}\int_0^{\infty}\zeta|\hat{u}(\zeta)|^2 d\zeta = \frac{1}{2\pi}\|u\|^2_{\dot{H}^{1/2}}.\end{aligned}$$

□

Lemma 3.6. *$\operatorname{Ker} H_u^2 = \operatorname{Ker} H_u$. Moreover, if $\operatorname{Ran} H_u$ is finite dimensional, then $\operatorname{Ran} H_u^2 = \operatorname{Ran} H_u$.*

Proof. Let $f \in \operatorname{Ker} H_u^2$. Then, by (1.7),

$$(H_u(h_1), h_2) = (H_u(h_2), h_1) \text{ for all } h_1, h_2 \in L^2_+,$$

we have

$$\|H_u f\|^2_{L^2} = (H_u f, H_u f) = (H_u^2 f, f) = 0$$

and thus $H_u f = 0$. Hence, $\operatorname{Ker} H_u^2 \subset \operatorname{Ker} H_u$. Therefore, we obtain $\operatorname{Ker} H_u^2 = \operatorname{Ker} H_u$ since the inverse inclusion is obvious.

The identity (1.7) yields also $\operatorname{Ker} H_u = (\operatorname{Ran} H_u)^\perp$. Moreover, it implies that H_u^2 is a self-adjoint operator and therefore, $\operatorname{Ker} H_u^2 = (\operatorname{Ran} H_u^2)^\perp$. Hence, we obtain

$$(\operatorname{Ran} H_u^2)^\perp = (\operatorname{Ran} H_u)^\perp.$$

Taking the orthogonal complement of both sides, this yields

$$\overline{\operatorname{Ran} H_u^2} = \overline{\operatorname{Ran} H_u.}$$

If $\operatorname{Ran} H_u$ is finite dimensional, so is $\operatorname{Ran} H_u^2$, since $\operatorname{Ran} H_u^2 \subset \operatorname{Ran} H_u$. Thus, $\operatorname{Ran} H_u^2$ and $\operatorname{Ran} H_u$ are closed. Hence, we have $\operatorname{Ran} H_u^2 = \operatorname{Ran} H_u$. □

Lemma 3.7. *If u is a traveling wave, then*

$$A_u H_u^2 = H_u^2 A_u. \tag{3.3}$$

Consequently, if $\operatorname{Ran} H_u$ is finite dimensional, then $A_u(\operatorname{Ran} H_u) \subset \operatorname{Ran} H_u$.

Proof. The commutativity relation (3.3) is a consequence of identity (1.10). The second statement then follows by Lemma 3.6, $\operatorname{Ran} H_u^2 = \operatorname{Ran} H_u$. □

It is a classical fact that if A and B are two self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $AB = BA$, then $B\big(\mathcal{H}_{\mathrm{ac}}(A)\big) \subset \mathcal{H}_{\mathrm{ac}}(A)$. For the sake of completeness, we prove it here in the case of the operators A_u and H_u^2.

Lemma 3.8. *$H_u^2 \mathcal{H}_{\mathrm{ac}}(A_u) \subset \mathcal{H}_{\mathrm{ac}}(A_u)$.*

Proof. As we see below, the inclusion follows if we prove that $\mu_{H_u^2\phi} \ll \mu_\phi$ for all $\phi \in L^2_+$, where the measures above are the spectral measures with respect to the operator A_u, corresponding respectively to $H_u^2\phi$ and ϕ.

Let $E \subset \mathbb{R}$ be a measurable set and $f = \mathbf{1}_E$. Then, by (3.3) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\mu_{H_u^2\phi}(E) &= \int_{\mathbb{R}} f d\mu_{H_u^2\phi} = (H_u^2\phi, f(A_u)H_u^2\phi) \\
&= (H_u^2\phi, H_u^2 f(A_u)\phi) = (H_u^4\phi, f(A_u)\phi) \\
&\le \sqrt{(f(A_u)\phi, f(A_u)\phi)}\|H_u^4\phi\|_{L^2} = \sqrt{(\phi, f(A_u)\phi)}\|H_u^4\phi\|_{L^2} \\
&= \sqrt{\mu_\phi(E)}\|H_u^4\phi\|_{L^2}.
\end{aligned}$$

Therefore, $\mu_{H_u^2\phi} \ll \mu_\phi$.

Let us denote by m the Lebesgue measure on $\mathbb{R}$. If $\phi \in \mathcal{H}_{\mathrm{ac}}(A_u)$, then $\mu_\phi \ll m$ and thus $\mu_{H_u^2\phi} \ll m$. Hence, $H_u^2\mathcal{H}_{\mathrm{ac}}(A_u) \subset \mathcal{H}_{\mathrm{ac}}(A_u)$. □

Proposition 3.9. *If u is a traveling wave, then $\mathcal{H}_{\mathrm{ac}}(A_u) \subset \operatorname{Ker} H_u$.*

Proof. It is enough to prove that $\left[\Omega^+(D, A_u)H_u^2\right]\left(\mathcal{H}_{\mathrm{ac}}(A_u)\right) = 0$. If this holds, then we have $H_u^2\left(\mathcal{H}_{\mathrm{ac}}(A_u)\right) = 0$ since $H_u^2\mathcal{H}_{\mathrm{ac}}(A_u) \subset \mathcal{H}_{\mathrm{ac}}(A_u)$ and $\Omega^+(D, A_u)$ is an isometry on $\mathcal{H}_{\mathrm{ac}}(A_u)$. Therefore, $\mathcal{H}_{\mathrm{ac}}(A_u) \subset \operatorname{Ker} H_u^2 = \operatorname{Ker} H_u$.

Let us first note that

$$H_u e^{itD} = e^{itD} H_{\tau_t(u)}, \tag{3.4}$$

where τ_a denotes the translation $\tau_a u(x) = u(x-a)$. Indeed, for $f \in L^2_+$, passing into the Fourier space, we have

$$\begin{aligned}\left(H_u e^{itD} f\right)^\wedge(\xi) &= \mathbf{1}_{\xi\geq 0}\left(u\overline{e^{itD} f}\right)^\wedge(\xi) = \frac{1}{2\pi}\mathbf{1}_{\xi\geq 0}\int_{\mathbb{R}} \hat{u}(\xi-\eta)e^{it\eta}\hat{\bar{f}}(\eta)d\eta \\ &= \frac{1}{2\pi}\mathbf{1}_{\xi\geq 0}e^{it\xi}\int e^{-it(\xi-\eta)}\hat{u}(\xi-\eta)\hat{\bar{f}}(\eta)d\eta = \mathbf{1}_{\xi\geq 0}e^{it\xi}\left(\tau_t(u)\bar{f}\right)^\wedge(\xi) \\ &= \mathbf{1}_{\xi\geq 0}\left(e^{itD}(\tau_t(u)\bar{f})\right)^\wedge(\xi) = \left(e^{itD}H_{\tau_t(u)}f\right)^\wedge(\xi).\end{aligned}$$

By Lemma 3.8, (3.3), and (3.4), we have for all $f \in \mathcal{H}_{\mathrm{ac}}(A_u)$

$$\begin{aligned}e^{itD}e^{-itA_u}P_{\mathrm{ac}}H_u^2 f &= e^{itD}e^{-itA_u}H_u^2 f = e^{itD}H_u^2 e^{-itA_u} f = e^{itD}H_u H_u e^{-itD}e^{itD}e^{-itA_u} f \\ &= e^{itD}H_u e^{-itD}H_{\tau_{-t}(u)}e^{itD}e^{-itA_u} f = H^2_{\tau_{-t}(u)}e^{itD}e^{-itA_u}P_{\mathrm{ac}}(A_u)f.\end{aligned}$$

We intend to prove that $H^2_{\tau_{-t}(u)}e^{itD}e^{-itA_u}P_{\mathrm{ac}}(A_u)f$ tends to 0 in the L^2_+-norm as $t \to -\infty$. From this, we conclude that $\Omega^+(D, A_u)H_u^2 f = 0$.

Since, by Lemma 3.5, $H_{\tau_{-t}(u)}$ is a uniformly bounded operator, it is enough to prove that $H_{\tau_{-t}(u)}e^{itD}e^{-itA_u}P_{\mathrm{ac}}(A_u)f$ tends to 0.

$$\begin{aligned}&\|H_{\tau_{-t}(u)}e^{itD}e^{-itA_u}P_{\mathrm{ac}}(A_u)f\|_{L^2_+} \\ &\qquad\leq \left\|H_{\tau_{-t}(u)}\Big(e^{itD}e^{-itA_u}P_{\mathrm{ac}}(A_u)f - \Omega^+(D,A_u)f\Big)\right\|_{L^2_+} \\ &\qquad\qquad + \|H_{\tau_{-t}(u)}\Omega^+(D,A_u)f\|_{L^2_+} \\ &\qquad\leq \frac{1}{\sqrt{2\pi}}\|u\|_{\dot{H}^{1/2}}\|e^{itD}e^{-itA_u}P_{\mathrm{ac}}(A_u)f - \Omega^+(D,A_u)f\|_{L^2_+} \\ &\qquad\qquad + \int_{\mathbb{R}}|u(x+t)|^2|\Omega^+(D,A_u)f(x)|^2dx\end{aligned} \tag{3.5}$$

The first term in (3.5) converges to 0 by the definition of the wave operator $\Omega^+(D, A_u)$.

Since u is a traveling wave,

$$u \in \bigcap_{s\geq 0} H^s(\mathbb{R}) \subset C^\infty_{\to 0}(\mathbb{R}),$$

where $C^\infty_{\to 0}(\mathbb{R})$ is the space of functions f of class C^∞ such that $\lim_{x\to-\infty} D^k f(x) = \lim_{x\to\infty} D^k f(x) = 0$ for all $k \in \mathbb{N}$. Therefore, for arbitrary fixed x, we have

$$\lim_{t\to-\infty}\tau_{-t}(u)(x) = \lim_{t\to-\infty}u(x+t) = 0.$$

Note also that $|u(x+t)|^2|\Omega^+(D,A_u)f(x)|^2 \leq \|u\|_{L^\infty}|\Omega^+(D,A_u)f(x)|^2$ for all $x \in \mathbb{R}$. Then the second term in (3.5) converges to 0 by the dominated convergence theorem. Hence $\left[\Omega^+(D,A_u)H_u^2\right]\left(\mathcal{H}_{\mathrm{ac}}(A_u)\right) = 0$. □

4. Classification of traveling waves

Lemma 4.1. *There are no nontrivial traveling waves of velocity $c = 0$ in $L^2_+(\mathbb{R})$.*

Proof. Let u be a nontrivial traveling wave of velocity $c = 0$. Then, equation 1.4 gives $\Pi(|u|^2 u) = \omega u$. Taking the scalar product with $e^{i\xi x}u(x)$, where $\xi \geq 0$, we obtain

$$\mathcal{F}(|u|^4 - \omega|u|^2)(\xi) = 0,$$

where $\mathcal{F}$ denotes the Fourier transform. Since $|u|^4 - \omega|u|^2$ is a real valued function, we have that the last equality holds for all $\xi \in \mathbb{R}$. Thus $|u|^4 - \omega|u|^2 = 0$ on $\mathbb{R}$ and therefore $u(x) = 0$ or $|u(x)|^2 = \omega > 0$, for all $x \in \mathbb{R}$. Since the function u is holomorphic on $\mathbb{C}_+$, its trace on $\mathbb{R}$ is either identically zero, or the set of zeros of u on $\mathbb{R}$ has Lebesgue measure zero. In conclusion, we have $|u|^2 = \omega > 0$ a.e. on $\mathbb{R}$ and thus u is not a function in $L^2_+(\mathbb{R})$. □

Lemma 4.2. *If $u \in H^s_+$ for $s > \frac{1}{2}$ and $v \in \operatorname{Ker} H_u$, then $\bar{u}v \in L^2_+$. Moreover, if $u \in L^\infty(\mathbb{R})$, then $T_{|u|^2}v = |u|^2 v$.*

Proof. Indeed, $0 = H_u(v) = \Pi(u\bar{v})$ and thus $\bar{u}v \in L^2_+$. Furthermore, since $u, \bar{u}v \in L^2_+$, we obtain $T_{|u|^2}v = \Pi(u\bar{u}v) = |u|^2 v$. □

Lemma 4.3. *Let $u \in H^s_+$, $s > \frac{1}{2}$, be a solution of the cubic Szegö equation* (1.2). *Consider the following Cauchy problem:*

$$\begin{cases} i\partial_t \psi = |u(t)|^2 \psi \\ \psi\big|_{t=0} = \psi_0, \end{cases} \tag{4.1}$$

If $\psi_0 \in \operatorname{Ker} H_{u(0)}$, then $\psi(t) \in \operatorname{Ker} H_{u(t)}$ for all $t \in \mathbb{R}$.

Proof. Let us first consider:

$$\begin{cases} i\partial_t \psi_1 = T_{|u(t)|^2} \psi_1 \\ \psi_1\big|_{t=0} = \psi_0, \end{cases}$$

Using the Lax pair structure, we have

$$\begin{aligned} \partial_t H_u(\psi_1) &= [B_u, H_u]\psi_1 + H_u \partial_t \psi_1 = [\frac{i}{2}H_u^2 - iT_{|u|^2}, H_u]\psi_1 + H_u(-iT_{|u|^2}\psi_1) \\ &= -iT_{|u|^2}H_u\psi_1 - iH_uT_{|u|^2}\psi_1 + iH_uT_{|u|^2}\psi_1 = -iT_{|u|^2}H_u\psi_1. \end{aligned}$$

The solution of this linear Cauchy problem

$$\begin{cases} \partial_t H_u(\psi_1) = -iT_{|u|^2}H_u\psi_1 \\ H_u(\psi_1(0)) = 0 \end{cases}$$

is identically zero. i.e., $H_{u(t)}\psi_1(t) = 0$ for all $t \in \mathbb{R}$. Consequently, $\psi_1(t) \in \operatorname{Ker} H_{u(t)}$ and by Lemma 4.2 we obtain $T_{|u|^2}\psi_1 = |u|^2\psi_1$. In conclusion, $\psi(t) = \psi_1(t) \in \operatorname{Ker} H_{u(t)}$. □

The space $\operatorname{Ker} H_u$ is invariant under multiplication by $e^{i\alpha x}$, for all $\alpha \geq 0$. Indeed, suppose $f \in \operatorname{Ker} H_u$. Then $\widehat{u\bar{f}}(\xi) = 0$, for all $\xi \geq 0$ and

$$\big(H_u(e^{i\alpha x}f)\big)^\wedge(\xi) = \big(e^{-i\alpha x}u\bar{f}\big)^\wedge(\xi) = \widehat{u\bar{f}}(\xi + \alpha) = 0,$$

for all $\xi, \alpha \geq 0$. Hence, $e^{i\alpha x}f \in \operatorname{Ker} H_u$ for all $\alpha \geq 0$.

One can then apply the following theorem to the subspaces $\operatorname{Ker} H_{u_0}$.

Proposition 4.4 (Lax [10]). *Every non-empty closed subspace of L^2_+ which is invariant under multiplication by $e^{i\alpha x}$ for all $\alpha \geq 0$ is of the form FL^2_+, where F is an analytic function in the upper-half plane, $|F(z)| \leq 1$ for all $z \in \mathbb{C}_+$, and $|F(x)| = 1$ for all $x \in \mathbb{R}$. Moreover, F is uniquely determined up to multiplication by a complex constant of absolute value 1.*

We deduce that $\operatorname{Ker} H_{u_0} = \phi L^2_+$, where ϕ is a holomorphic function in the upper half-plane $\mathbb{C}_+$, satisfying $|\phi(x)| = 1$ on $\mathbb{R}$ and $|\phi(z)| \leq 1$ for all $z \in \mathbb{C}_+$.

Functions satisfying the properties in Lax's theorem are called inner functions in the sense of Beurling-Lax. A special class of inner functions is given by the Blaschke products. Given $\lambda_j \in \mathbb{C}$ such that for all j

$$\operatorname{Im} \lambda_j > 0$$

and

$$\sum_j \frac{\operatorname{Im} \lambda_j}{1 + |\lambda_j|^2} < \infty,$$

the corresponding Blaschke product is defined by

$$B(z) = \prod_j \varepsilon_j \frac{z - \lambda_j}{z - \overline{\lambda}_j}, \tag{4.2}$$

where $\varepsilon_j = \frac{|\lambda_j^2+1|}{\lambda_j^2+1}$ (by definition $\varepsilon_j = 1$ if $\lambda_j = 1$).

Inner functions have a canonical factorization, which is analogous to the canonical factorization of inner functions on the unit disk, see [16, Theorem 17.15], [12, Theorem 6.4.4]. More precisely, every inner function F can be written as the product

$$F(z) = \lambda B(z) e^{iaz} e^{i \int_{\mathbb{R}} \frac{1+tz}{t-z} d\nu(t)}, \tag{4.3}$$

where $z \in \mathbb{C}_+$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, $a \geq 0$, B is a Blaschke product, and ν is a positive singular measure with respect to the Lebesgue measure. In particular, the inner function ϕ has such a canonical factorization.

Proposition 4.5. *Let u be a traveling wave and denote by ϕ an inner function such that* $\operatorname{Ker} H_{u_0} = \phi L^2_+$. *Then, ϕ satisfies the following equation on $\mathbb{R}$:*

$$cD\phi = |u_0|^2 \phi. \tag{4.4}$$

Proof. Since $u(t,x) = e^{-i\omega t} u_0(x - ct)$, we have $H_{u(t)} = e^{-i\omega t} \tau_{ct} H_{u_0} \tau_{-ct}$. Thus,

$$\operatorname{Ker} H_{u(t)} = \tau_{ct} \operatorname{Ker} H_{u_0} = \tau_{ct}(\phi) L^2_+.$$

Let $f \in L^2_+$ and let $\psi_0 = \phi f \in \operatorname{Ker} H_{u_0}$ be the initial data of the Cauchy problem (4.1) in Lemma 4.3. We then have $\phi e^{-i \int_0^t |u_s|^2 ds} f \in \operatorname{Ker} H_{u(t)}$. Therefore,

$$\phi e^{-i \int_0^t |u_s|^2 ds} L^2_+ \subset \tau_{ct}(\phi) L^2_+. \tag{4.5}$$

Conversely, by solving backward the problem (4.1) with the initial data in $\tau_{ct}(\phi) L^2_+$ at time t, up to the time $t = 0$, we obtain

$$\tau_{ct}(\phi) L^2_+ \subset \phi e^{-i \int_0^t |u_s|^2 ds} L^2_+$$

and thus, the two sets are equal.

Let us first prove that $\phi_t := \phi e^{-i\int_0^t |u_s|^2 ds}$ is an inner function. Note that ϕ_t is well defined on $\mathbb{R}$ and its absolute value is 1 on $\mathbb{R}$. Consider the function defined by $h(x) = \frac{\phi_t(x)}{x+i}$, for all $x \in \mathbb{R}$. Since $h \in L^2_+$, we can write using the Poisson integral that

$$h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im} z \frac{h(x)}{|z-x|^2} dx,$$

for all $z \in \mathbb{C}_+$. Then,

$$zh(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im} z \frac{xh(x)}{|z-x|^2} dx + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im} z \frac{(z-x)h(x)}{|z-x|^2} dx.$$

Note that the last integral is equal to $\int_{-\infty}^{\infty} \mathrm{Im} z \frac{h(x)}{\bar{z}-x} dx$. By the residue theorem and using the fact that the function $\frac{h}{\bar{z}-x}$ is holomorphic on $\mathbb{C}_+$, we have that this integral is zero and thus

$$zh(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im} z \frac{xh(x)}{|z-x|^2} dx.$$

Therefore, we can use the Poisson integral to extend ϕ_t to $\mathbb{C}_+$ as a holomorphic function.

$$\phi_t(z) = (z+i)h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im} z \frac{(x+i)h(x)}{|z-x|^2} dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im} z \frac{\phi_t(x)}{|z-x|^2} dx. \tag{4.6}$$

Moreover,

$$|\phi_t(z)| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Im} z \frac{1}{|z-x|^2} dx = 1,$$

for all $z \in \mathbb{C}_+$. Hence ϕ_t is an inner function.

Since $\tau_{ct}(\phi)$ and $\phi e^{-i\int_0^t |u_s|^2 ds}$ are inner functions and

$$\phi e^{-i\int_0^t |u_s|^2 ds} L^2_+ = \tau_{ct}(\phi) L^2_+,$$

Proposition 4.4 yields the existence of a real valued function γ such that $\gamma(0) = 0$ and

$$\phi e^{-i\int_0^t |u_s|^2 ds} = \tau_{ct}(\phi) e^{i\gamma(t)}.$$

Taking the derivative with respect to t we obtain that ϕ satisfies the following equation:

$$cD\phi(x) = |u(t, x+ct)|^2 \phi(x) + \dot{\gamma}(t)\phi(x).$$

for all $t \in \mathbb{R}$. Since u is a traveling wave, we have $|u(t, x+ct)| = |e^{-i\omega t} u_0(x)| = |u_0(x)|$. Then we deduce that $\dot{\gamma}(t) = k$ and hence $\gamma(t) = kt$, for some $k \in \mathbb{R}$. Therefore,

$$cD\phi = (|u_0|^2 + k)\phi. \tag{4.7}$$

We prove in the following that $k = 0$. First, note that $\frac{k}{c} \geq 0$. The function $\phi u_0 \in \mathrm{Ker} H_{u_0}$ and by Lemma 4.2, we have $|u_0|^2\phi = \overline{u}_0(u_0\phi) \in L^2_+$. If $\frac{k}{c}$ is negative, denoting $\chi := \frac{1}{c}|u_0|^2\phi \in L^2_+$ and passing into the Fourier space, we have:

$$\hat{\phi}(\xi) = \frac{1}{\xi - \frac{k}{c}} \hat{\chi}(\xi) \mathbf{1}_{[0,\infty)}(\xi).$$

This implies that $\phi \in L^2_+$, contradicting $|\phi(x)| = 1$ for all $x \in \mathbb{R}$.

Let us now prove that $\frac{k}{c} = 0$. Let $h \in L^2_+$ regular. Then $\phi h \in \mathrm{Ker}\, H_{u_0}$ and by equation (4.7) we have

$$A_{u_0}(\phi h) = (D - \tfrac{1}{c}|u_0|^2)(\phi h) = \phi(D - \tfrac{1}{c}|u_0|^2)(h) + hD\phi = \phi(D + \tfrac{k}{c})h.$$

Denoting by $\mu_{\phi h}(A_{u_0})$ the spectral measure corresponding to ϕh, we have

$$\int f d\mu_{\phi h} = (\phi h, f(A_{u_0})\phi h) = (\phi h, \phi f(D+\tfrac{k}{c})h) = (h, f(D+\tfrac{k}{c})h)$$
$$= \frac{1}{2\pi}\int_0^\infty f(\xi+\tfrac{k}{c})|\hat{h}(\xi)|^2 d\xi = \frac{1}{2\pi}\int_{\frac{k}{c}}^\infty f(\eta)|\hat{h}(\eta-\tfrac{k}{c})|^2 d\eta.$$

Consequently, $\operatorname{supp}\mu_{\phi h}(A_{u_0}) \subset [\frac{k}{c}, +\infty)$. By Proposition 3.9, we have $\mathcal{H}_{\mathrm{ac}}(A_{u_0}) \subset \operatorname{Ker} H_{u_0}$, and therefore

$$\sigma_{\mathrm{ac}}(A_{u_0}) = \overline{\bigcup_{\psi\in\mathcal{H}_{\mathrm{ac}}(A_{u_0})} \operatorname{supp}\mu_\psi} \subset \overline{\bigcup_{\phi h\in\operatorname{Ker}H_{u_0}} \operatorname{supp}\mu_{\phi h}} \subset \big[\tfrac{k}{c}, \infty\big).$$

Since, by Corollary 3.4, $\sigma_{\mathrm{ac}}(A_{u_0}) = [0,\infty)$, this yields $k=0$. □

Proposition 4.6. *All traveling waves are rational fractions.*

Proof. We first prove that ϕ is a Blaschke product.

Since ϕ is an inner function in the sense of Beurling-Lax, it has the following canonical decomposition:

$$\phi(z) = \lambda B(z) e^{iaz} e^{i\int_{\mathbb{R}} \frac{1+tz}{t-z} d\nu(t)}, \tag{4.8}$$

where $z\in\mathbb{C}_+$, λ is a complex number of absolute value 1, $a\geq 0$, B is a Blaschke product having exactly the same zeroes as ϕ, and ν is a positive singular measure with respect to the Lebesgue measure.

Because ϕ satisfies the equation (4.4) and $u_0\in L^\infty(\mathbb{R})$, we obtain that ϕ has bounded derivative on $\mathbb{R}$ and hence it is uniformly continuous on $\mathbb{R}$. Then, since ϕ satisfies the Poisson formula (4.6), it follows that

$$\phi(x+i\varepsilon) \to \phi(x), \text{ as } \varepsilon\to 0,$$

uniformly for $x\in\mathbb{R}$.

ϕ being uniformly continuous on $\mathbb{R}$ and $|\phi(x)|=1$, $\forall x\in\mathbb{R}$, we deduce that the zeroes of ϕ and hence, those of the Blaschcke product B as well, lie outside a strip $\{z\in\mathbb{C}; 0\leq \operatorname{Im} z\leq\varepsilon_0\}$, for some $\varepsilon_0>0$. Therefore, we have

$$\frac{\phi(x+i\varepsilon)}{B(x+i\varepsilon)} \to \frac{\phi(x)}{B(x)}, \text{ as } \varepsilon\to 0$$

uniformly for x in compact subsets of $\mathbb{R}$. Taking the logarithm of the absolute value and noticing that $\big|\frac{\phi(x)}{B(x)}\big| = 1$, we obtain

$$\int_{\mathbb{R}} \frac{\varepsilon}{(x-t)^2+\varepsilon^2} d\nu(t) \to 0,$$

uniformly for x in compact subsets in $\mathbb{R}$. In particular, for all $\delta>0$ there exists $0<\varepsilon_1\leq\varepsilon_0$ such that for all $0<\varepsilon\leq\varepsilon_1$ and for all $x\in[0,1]$, we have

$$\frac{1}{2\varepsilon}\nu([x-\varepsilon, x+\varepsilon]) \leq \int_{x-\varepsilon}^{x+\varepsilon} \frac{\varepsilon}{(x-t)^2+\varepsilon^2} d\nu(t) \leq \int_{\mathbb{R}} \frac{\varepsilon}{(x-t)^2+\varepsilon^2} d\nu(t) \leq \delta.$$

Taking $\varepsilon = \frac{1}{2N} \leq \varepsilon_1$ with $N\in\mathbb{N}^*$, we obtain

$$\nu([0,1]) = \nu\Big(\bigcup_{k=0}^{N-1}[\frac{k}{N}, \frac{k+1}{N}]\Big) \leq N\delta\frac{1}{N} = \delta.$$

In conclusion $\nu([0,1]) = 0$, and one can prove similarly that the measure ν of any compact interval in $\mathbb{R}$ is zero. Hence $\nu \equiv 0$.

Consequently, $\phi(x) = \lambda B(x) e^{iax}$ for all $x \in \mathbb{R}$. On the other hand, because ϕ satisfies the equation (4.4), we have $\phi(x) = \phi(0) e^{\frac{i}{c}\int_0^x |u_0|^2}$ and, in particular, $\lim_{x\to\infty}\phi(x) = \phi(0) e^{\frac{i}{c}\int_0^\infty |u_0|^2}$. Since $\lim_{x\to\infty} B(x) = 1$, we conclude that $a = 0$. Substituting $\phi = \lambda B$ in the equation (4.4), we obtain

$$\frac{c}{i}\frac{B'}{B} = |u_0|^2.$$

Then $\frac{1}{i}\int_{-\infty}^{\infty} \frac{B'(x)}{B(x)}\, dx < \infty$. Computing this integral, we obtain that

$$\frac{1}{i}\int_{-\infty}^{\infty} \frac{B'(x)}{B(x)}\, dx = 2\sum_j \int_{-\infty}^{\infty} \frac{\mathrm{Im}\lambda_j}{|x-\lambda_j|^2}\, dx = 2\sum_j \pi$$

and thus it is finite if and only if B is a finite Blaschke product, $B(x) = \prod_{j=1}^N \varepsilon_j \frac{x-\lambda_j}{x-\overline{\lambda}_j}$.

Let us prove that the traveling wave u is a rational fraction.

$$\operatorname{Ker} H_u = \phi L^2_+ = B L^2_+.$$

Notice that $BL^2_+ = \left(\mathrm{span}_{\mathbb{C}}\left\{\frac{1}{x-\overline{\lambda}_j}\right\}_{j=1}^N\right)^{\perp}$. Indeed, $f \in \left(\mathrm{span}_{\mathbb{C}}\left\{\frac{1}{x-\overline{\lambda}_j}\right\}_{j=1}^N\right)^{\perp}$ if and only if

$$f(\lambda_j) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi\lambda_j}\widehat{f}(\xi)d\xi = \frac{1}{2\pi}\left(\hat{f}, e^{-i\overline{\lambda}_j\xi}\right) = \left(f, \frac{1}{x-\overline{\lambda}_j}\right) = 0,$$

if and only if there exists $h \in L^2_+$ such that $f = Bh$. Hence

$$\operatorname{Ker} H_u = \left(\mathrm{span}_{\mathbb{C}}\left\{\frac{1}{x-\overline{\lambda}_j}\right\}_{j=1}^N\right)^{\perp}$$

This yields $\overline{\operatorname{Ran} H_u} = \mathrm{span}_{\mathbb{C}}\left\{\frac{1}{x-\overline{\lambda}_j}\right\}_{j=1}^N$. By Remark 2.2 it follows that u is a rational fraction. More precisely, $u \in \operatorname{Ran} H_u = \mathrm{span}_{\mathbb{C}}\left\{\frac{1}{x-\overline{\lambda}_j}\right\}_{j=1}^N$. □

Proposition 4.7. *If u is a traveling wave, then there exists $\lambda > 0$ such that $H_u^2 u = \lambda u$.*

Proof. According to Remark 2.2, since u is a rational fraction, we have $u \in \operatorname{Ran} H_u$.

Secondly, u satisfies the equation of the traveling waves (1.4), which is equivalent to $A_u(u) = -\frac{\omega}{c}u$. Therefore, u is an eigenfunction of the operator A_u for the eigenvalue $-\frac{\omega}{c}$. Applying the identity (1.10),

$$A_u H_u + H_u A_u + \frac{\omega}{c} H_u + \frac{1}{c} H_u^3 = 0,$$

to u and then to $H_u u$, one deduces that $A_u H_u^2 u = -\frac{\omega}{c} H_u^2 u$. Therefore, the conclusion of the proposition follows once we prove all the eigenfunctions of the operator A_u belonging to $\operatorname{Ran} H_u$, corresponding to the same eigenvalue, are linearly dependent.

Let a be en eigenvalue of the operator A_u and let $\psi_1, \psi_2 \in \operatorname{Ker}(A_u - a) \cap \operatorname{Ran} H_u$. Since u is a rational fraction, by the Kronecker type theorem 2.1, ψ_1 and ψ_2 are also nonconstant rational fractions. Then, one can find $\alpha, \beta \in \mathbb{C}$, $(\alpha,\beta) \neq (0,0)$, such that $\psi := \alpha\psi_1 + \beta\psi_2 = O(\frac{1}{x^2})$ as $x \to \infty$. Moreover, we have $\psi \in L^1(\mathbb{R})$, $x\psi \in L^2(\mathbb{R})$, and thus we can compute $A_u(x\psi)$.

Passing into the Fourier space we have,

$$\widehat{\Pi(xf)}(\xi) = i(\partial_\xi \hat{f})\mathbf{1}_{\xi\geq 0} = i\partial_\xi(\hat{f}\mathbf{1}_{\xi\geq 0}) - i\hat{f}(\xi)\delta_{\xi=0} = \widehat{x\Pi f}(\xi) - i\hat{f}(0)\delta_{\xi=0},$$

for all $f \in L^1(\mathbb{R})$. Thus, we obtain $\Pi(xf) = x\Pi(f) + \frac{1}{2\pi i}\hat{f}(0)$ for all $f \in L^1(\mathbb{R})$. We then have

$$A_u(x\psi) = xA_u(\psi) + \frac{1}{i}\psi - \frac{1}{2c\pi i}\int_{\mathbb{R}} |u|^2\psi dx$$

and therefore, since $A_u\psi = a\psi$,

$$A_u(x\psi) = ax\psi + \frac{1}{i}\psi - \frac{1}{2c\pi i}\int_{\mathbb{R}} |u|^2\psi dx. \tag{4.9}$$

Since $x\psi \in \operatorname{Ran} H_u$ and $A_u(\operatorname{Ran} H_u) \subset \operatorname{Ran} H_u$ by Lemma 3.7, we have $A_u(x\psi) \in \operatorname{Ran} H_u \subset L^2(\mathbb{R})$. The constant in equation (4.9) is zero because all the other terms are in $L^2(\mathbb{R})$. Then we have

$$(A_u - a)(x\psi) = \frac{1}{i}\psi. \tag{4.10}$$

Applying the self-adjoint operator $A_u - a$ to both sides of the equation (4.10), we obtain $(A_u - a)^2(x\psi) = 0$ and

$$\|(A_u - a)(x\psi)\|_{L^2}^2 = ((A_u - a)(x\psi), (A_u - a)(x\psi)) = ((A_u - a)^2(x\psi), x\psi) = 0.$$

Thus, $(A_u - a)(x\psi) = 0$. In conclusion, by equation (4.10), $\psi = 0$ and therefore all the eigenfunctions belonging to $\operatorname{Ran} H_u$, corresponding to the same eigenvalue a, are linearly dependent. □

Proof of Theorem 1.2. Since $u \in \operatorname{Ran} H_u$, there exists a unique function $g \in \operatorname{Ran} H_u$ such that $u = H_u(g)$. By Lemma 4.7, it results that $H_u(u) = \lambda g$. Applying the identity (1.10),

$$A_u H_u + H_u A_u + \frac{\omega}{c}H_u + \frac{1}{c}H_u^3 = 0,$$

to g and using $A_u u = -\frac{\omega}{c}u$, one obtains $H_u(A_u g + \frac{\lambda}{c}g) = 0$. Since $A_u(\operatorname{Ran} H_u) \subset \operatorname{Ran} H_u$, we have $A_u g + \frac{\lambda}{c}g \in \operatorname{Ran} H_u \cap \operatorname{Ker} H_u$. Therefore, $A_u g + \frac{\lambda}{c}g = 0$, which is equivalent to

$$cDg - T_{|u|^2}g + \lambda g = 0.$$

In the following we intend to find a simpler version of the above equation, in order to determine the function g explicitely. Note that $\bar{u}(1-g) \in L^2_+$, since it is orthogonal to each complex conjugate of a holomorphic function $f \in L^2_+$:

$$(\bar{u}(1-g), \bar{f}) = (f(1-g), u) = (f, u) - (f, H_u(g)) = 0.$$

Thus, $T_{|u|^2}(g) = \Pi(|u|^2) - \Pi(|u|^2(1-g)) = H_u(u) - |u|^2(1-g) = \lambda g - |u|^2(1-g)$.

Passing into the Fourier space and using the fact that $|u|^2$ is a real valued function, one can write

$$|u|^2 = \int_0^\infty e^{ix\xi}\widehat{|u|^2}(\xi)d\xi + \int_0^\infty e^{-ix\xi}\overline{\widehat{|u|^2}}(\xi)d\xi = \Pi(|u|^2) + \overline{\Pi(|u|^2)}.$$

Therefore $|u|^2 = H_u(u) + \overline{H_u(u)} = \lambda(g + \overline{g})$. Consequently, $T_{|u|^2}(g) = \lambda(-\bar{g} + g^2 + |g|^2)$ and g solves the equation

$$cDg - \lambda g^2 + \lambda(g + \bar{g} - |g|^2) = 0. \tag{4.11}$$

We prove that $g+\bar{g}-|g|^2=0$. First, note that $\bar{u}(1-g)\in L^2_+$, also yields $(1-g)f\in \operatorname{Ker} H_u$, for all $f\in L^2_+$. Secondly, let us prove that $g+\bar{g}-|g|^2$ is orthogonal to the complex conjugate of all $f\in L^2_+$:

$$(g+\bar{g}-|g|^2,\bar{f})=(g,\bar{f})-(f(1-g),g)=-(f(1-g),\tfrac{1}{\lambda}H_u(u))=-\tfrac{1}{\lambda}(u,H_u(f(1-g)))=0.$$

In addition, since $g+\bar{g}-|g|^2$ is a real valued function, we have

$$(g+\bar{g}-|g|^2,f)=(g+\bar{g}-|g|^2,\bar{f})=0$$

for all $f\in L^2_+$. Therefore, $g+\bar{g}-|g|^2$ is orthogonal to all the functions in $L^2(\mathbb{R})$ and thus $g+\bar{g}-|g|^2=0$. This is equivalent to $|1-g|=1$ on $\mathbb{R}$. Moreover, equation (4.11) gives the precise formula for g,

$$g(z)=\frac{r}{z-p},$$

where $r,p\in\mathbb{C}$ and $\operatorname{Im}(p)<0$. Thus $1-g(x)=\frac{x-\bar{p}}{x-p}$ for all $x\in\mathbb{R}$ and

$$\operatorname{Ker} H_{\frac{1}{z-p}}=\frac{z-\bar{p}}{z-p}L^2_+=(1-g)L^2_+\subset \operatorname{Ker} H_u.$$

Consequently, $u\in \operatorname{Ran} H_u\subset \operatorname{Ran} H_{\frac{1}{z-p}}=\frac{\mathbb{C}}{z-p}$. □

5. Orbital stability of traveling waves

In order to prove the orbital stability of traveling waves, we first use the fact that they are minimizers of the Gagliardo-Nirenberg inequality. We begin this section by proving this inequality, more precisely proposition 1.5.

Proof of Proposition 1.5, Gagliardo-Nirenberg inequality. The proof is similar to the proof of Gagliardo-Nirenberg inequality for the circle, in [6]. The idea is to write all the norms in the Fourier space, using Plancherel's identity.

$$E=\|u\|^4_{L^4}=\|u^2\|^2_{L^2}=\frac{1}{2\pi}\|\widehat{u^2}\|^2_{L^2}=\frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{u^2}(\xi)|^2d\xi.$$

Using the fact that $u\in L^2_+$ and Cauchy-Schwarz inequality, we have:

$$\begin{aligned}|\widehat{u^2}(\xi)|^2&=\frac{1}{4\pi^2}\Big|\int_0^\xi \widehat{u}(\eta)\widehat{u}(\xi-\eta)d\eta\Big|^2\le\frac{1}{4\pi^2}\xi\int_0^\xi|\widehat{u}(\eta)|^2|\widehat{u}(\xi-\eta)|^2d\eta\\&\le\frac{1}{4\pi^2}\Big(\int_0^\xi\eta|\widehat{u}(\eta)|^2|\widehat{u}(\xi-\eta)|^2d\eta+\int_0^\xi(\xi-\eta)|\widehat{u}(\eta)|^2|\widehat{u}(\xi-\eta)|^2d\eta\Big).\end{aligned}$$

By change of variables $\xi-\eta\mapsto\eta$ in the second integral, we have

$$|\widehat{u^2}(\xi)|^2\le\frac{1}{2\pi^2}\int_0^\xi\eta|\widehat{u}(\eta)|^2|\widehat{u}(\xi-\eta)|^2d\eta.$$

By Fubini's theorem and change of variables $\zeta=\xi-\eta$ it results that

$$E\le\frac{1}{4\pi^3}\int_{\mathbb{R}}\int_0^\xi\eta|\widehat{u}(\eta)|^2|\widehat{u}(\xi-\eta)|^2d\eta d\xi=\frac{1}{4\pi^3}\int_0^{+\infty}\eta|\widehat{u}(\eta)|^2d\eta\int_0^{+\infty}|\widehat{u}(\zeta)|^2d\zeta=\frac{1}{\pi}MQ.$$

Moreover, equality holds if and only if we have equality in Cauchy-Schwarz inequality, i.e.

$$\widehat{u}(\xi)\widehat{u}(\eta)=\widehat{u}(\xi+\eta)\widehat{u}(0),$$

for all $\xi,\eta\ge0$. This is true if and only if $\widehat{u}(\xi)=e^{-ip\xi}\widehat{u}(0)$, for all $\xi\ge0$. Since $u\in H^{1/2}_+$, this yields $\operatorname{Im}(p)<0$ and $u(x)=\frac{C}{x-p}$, for some constant C. □

The second argument we use in proving stability of traveling waves is a profile decomposition theorem. It states that bounded sequences in $H^{1/2}_+$ can be written as superposition of translations of fixed profiles and of a remainder term. The remainder is small in all the L^p-norms, $2 < p < \infty$. Moreover, the superposition is almost orthogonal in the $H^{1/2}_+$-norm.

Proposition 5.1 (The profile decomposition theorem for bounded sequences in $H^{1/2}_+$)**.** *Let $\{v^n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^{1/2}_+$. Then, there exist a subsequence of $\{v^n\}_{n\in\mathbb{N}}$, still denoted by $\{v^n\}_{n\in\mathbb{N}}$, a sequence of fixed profiles in $H^{1/2}_+$, $\{V^{(j)}\}_{j\in\mathbb{N}}$, and a family of real sequences $\{x^{(j)}\}_{j\in\mathbb{N}}$ such that for all $\ell\in\mathbb{N}^*$ we have*

$$v^n = \sum_{j=1}^{\ell} V^{(j)}(x - x_n^{(j)}) + r_n^{(\ell)},$$

where

$$\lim_{\ell\to\infty}\limsup_{n\to\infty} \|r_n^{(\ell)}\|_{L^p(\mathbb{R})} = 0$$

for all $p\in(2,\infty)$, and

$$\begin{aligned}
\|v^n\|^2_{L^2} &= \sum_{j=1}^{\ell} \|V^{(j)}\|^2_{L^2} + \|r_n^{(\ell)}\|^2_{L^2} + o(1), \quad \textit{as } n\to\infty,\\
\|v^n\|^2_{\dot H^{1/2}_+} &= \sum_{j=1}^{\ell} \|V^{(j)}\|^2_{\dot H^{1/2}_+} + \|r_n^{(\ell)}\|^2_{\dot H^{1/2}_+} + o(1), \quad \textit{as } n\to\infty,\\
\lim_{n\to\infty}\|v^n\|^4_{L^4} &= \sum_{j=1}^{\infty} \|V^{(j)}\|^4_{L^4}.
\end{aligned}$$

The proof of this proposition follows exactly the same lines as that of the profile decomposition theorem for bounded sequences in $H^1(\mathbb{R})$, [7, Proposition 2.1]. However, note that in our case, the profiles $V^{(j)}$ belong to the space $H^{1/2}_+$, (not only to the space $H^{1/2}(\mathbb{R})$), as they are weak limits of translations of the sequence $\{v^n\}_{n\in\mathbb{N}}$.

Proof of Corollary 1.3. According to Proposition 1.5, $C(a,r)$ is the set of minimizers of the problem

$$\inf\{M(u)\big| u\in H^{1/2}_+, Q(u) = q(a,r), E(u) = e(a,r)\},$$

where

$$q(a,r) = \frac{a^2\pi}{r}, \qquad e(a,r) = \frac{a^4\pi}{2r^3}.$$

We denote the infimum by $m(a,r)$.

Since

$$\inf_{\phi\in C(a,r)} \|u_0^n - \phi\|_{H^{1/2}_+} \to 0,$$

by the Sobolev embedding theorem, we deduce

$$Q(u_0^n)\to q(a,r), \qquad E(u_0^n)\to e(a,r), \qquad M(u_0^n)\to m(a,r).$$

Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. The conservation laws yield

$$Q(u^n(t_n))\to q(a,r), \quad E(u^n(t_n))\to e(a,r), \quad M(u^n(t_n))\to m(a,r).$$

We can choose two sequences of positive numbers $\{a_n\}$ and $\{\lambda_n\}$ such that $v^n(x) := a_n u^n(t_n, \lambda_n x)$ satisfies $\|v^n\|_{L^2(\mathbb{R})} = 1$, $\|v^n\|_{L^4(\mathbb{R})} = 1$. Notice that

$$a_n \to a_\infty, \quad \lambda_n \to \lambda_\infty,$$

where $a_\infty > 0$, $\lambda_\infty > 0$, and

$$\frac{\lambda_\infty}{a_\infty^4} = e(a,r), \quad \frac{\lambda_\infty}{a_\infty^2} = q(a,r).$$

Then

$$\|v^n\|_{\dot{H}^{1/2}_+}^{1/2} = \frac{\|v^n\|_{L^2}^{1/2}\|v^n\|_{\dot{H}^{1/2}_+}^{1/2}}{\|v^n\|_{L^4}} = \frac{\|u^n(t_n)\|_{L^2}^{1/2}\|u^n(t_n)\|_{\dot{H}^{1/2}_+}^{1/2}}{\|u^n(t_n)\|_{L^4}},$$

for all $n \in \mathbb{N}$. In particular, as a consequence of the Gagliardo-Nirenberg inequality,

$$\lim_{n\to\infty} \|v^n\|_{\dot{H}^{1/2}_+} = \sqrt{\pi}.$$

Thus, the sequence $\{v^n\}_{n\in\mathbb{N}}$ is bounded in $H^{1/2}_+$. Applying the profile decomposition theorem (Proposition 5.1), we obtain that there exist real sequences $\{x^{(j)}\}_{j\in\mathbb{N}}$ depending on the sequence $\{t_n\}_{n\in\mathbb{N}}$ in the definition of $\{v^n\}_{n\in\mathbb{N}}$, such that for all $\ell \in \mathbb{N}^*$ we have:

$$v^n = \sum_{j=1}^{\ell} V^{(j)}(x - x_n^{(j)}) + r_n^{(\ell)},$$

where

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \|r_n^{(\ell)}\|_{L^p(\mathbb{R})} = 0$$

for all $p \in (2, \infty)$, and

$$\|v^n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V^{(j)}\|_{L^2}^2 + \|r_n^{(\ell)}\|_{L^2}^2 + o(1), \quad \text{as } n \to \infty,$$

$$\|v^n\|_{\dot{H}^{1/2}_+}^2 = \sum_{j=1}^{\ell} \|V^{(j)}\|_{\dot{H}^{1/2}_+}^2 + \|r_n^{(\ell)}\|_{\dot{H}^{1/2}_+}^2 + o(1), \quad \text{as } n \to \infty,$$

$$\lim_{n\to\infty} \|v^n\|_{L^4}^4 = \sum_{j=1}^{\infty} \|V^{(j)}\|_{L^4}^4.$$

Consequently,

$$1 \geq \sum_{j=1}^{\infty} \|V^{(j)}\|_{L^2}^2, \qquad \pi \geq \sum_{j=1}^{\infty} \|V^{(j)}\|_{\dot{H}^{1/2}_+}^2, \qquad 1 = \sum_{j=1}^{\infty} \|V^{(j)}\|_{L^4}^4. \tag{5.1}$$

Therefore, by the Gagliardo-Nirenberg inequality (1.12), we have

$$\pi \geq (\sum_{j=1}^{\infty} \|V^{(j)}\|_{L^2}^2)(\sum_{j=1}^{\infty} \|V^{(j)}\|_{\dot{H}^{1/2}_+}^2) \geq \sum_{j=1}^{\infty} \|V^{(j)}\|_{L^2}^2 \|V^{(j)}\|_{\dot{H}^{1/2}_+}^2 \geq \pi \sum_{j=1}^{\infty} \|V^{(j)}\|_{L^4}^4 = \pi.$$

Thus, there exist only one profile $V := V^{(1)}$ and a sequence $x = x^{(1)}$ such that

$$v^n = V(x - x_n) + r_n,$$

$$\|v^n\|_{L^2}^2 = \|V\|_{L^2}^2 + \|r_n\|_{L^2}^2 + o(1), \quad \text{as } n \to \infty, \tag{5.2}$$

$$\|v^n\|_{\dot{H}_+^{1/2}}^2 = \|V\|_{\dot{H}_+^{1/2}}^2 + \|r_n\|_{\dot{H}_+^{1/2}}^2 + o(1), \quad \text{as } n \to \infty. \tag{5.3}$$

According to (5.1), V satisfies $1 \geq \|V\|_{L^2}^2$, $\pi \geq \|V\|_{\dot{H}_+^{1/2}}^2$, and $\|V\|_{L^4}^4 = 1$. In conclusion,

$$\pi = \pi\|V\|_{L^4}^4 \leq \|V\|_{L^2}^2 \|V\|_{\dot{H}_+^{1/2}}^2 \leq \pi.$$

Hence, V is a minimizer in the Gagliardo-Nirenberg inequality. Moreover,

$$\|V\|_{L^2}^2 = 1 = \|v^n\|_{L^2}, \qquad \|V\|_{\dot{H}_+^{1/2}}^2 = \pi = \lim_{n\to\infty} \|v^n\|_{\dot{H}_+^{1/2}}^2,$$

By (5.2) and (5.3), we have $r_n \to 0$ in $H_+^{1/2}$ as $n \to \infty$. Consequently, $v^n(\cdot + x_n) \to V$ in $H_+^{1/2}$, or equivalently,

$$\lim_{n\to\infty} \|a_n u^n(t_n, \lambda_n x) - V(x - x_n)\|_{H_+^{1/2}} = 0.$$

We then have

$$\lim_{n\to\infty} \|u^n(t_n, x) - \frac{1}{a_\infty} V(\frac{x - x_n\lambda_\infty}{\lambda_\infty})\|_{H_+^{1/2}} = 0.$$

Notice that, since V is a minimizer in the Gagliardo-Nirenberg inequality, we have $\tilde{\phi}(x) := \frac{1}{a_\infty} V(\frac{x}{\lambda_\infty}) = \frac{\alpha}{x-p} \in C(a, r)$. Then, since $x_n\lambda_\infty \in \mathbb{R}$, we have $\phi(x) = \tilde{\phi}(x - x_n\lambda_\infty) = \frac{\alpha}{x-\tilde{p}} \in C(a, r)$. Thus,

$$\inf_{\phi\in C(a,r)} \|u^n(t_n, x) - \phi(x)\|_{H_+^{1/2}} \to 0, \text{ as } n \to \infty. \tag{5.4}$$

The conclusion follows by approximating the supremum in the statement by the sequence in (5.4) with an appropriate $\{t_n\}_{n\in\mathbb{N}}$. □

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Oana Pocovnicu, Laboratoire de Mathématiques d'Orsay, Université Paris-Sud (XI), 91405, Orsay Cedex, France

E-mail address: `oana.pocovnicu@math.u-psud.fr`